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# STUDENTS' RESOLUTIONS OF SOME PARADOXES OF INFINITY IN THE LENS OF THE GROSSONE METHODOLOGY

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## Abstract

In the article, students' responses to some classical paradoxes related to infinity were investigated by considering the traditional methods and later by considering a new methodology for dealing with infinity (grossone methodology) which was introduced to them briefly. Eleven students of grade 12 participated in this study.

In phase 1, the students (who study in the traditional methods) provided their intuitive resolutions to three paradoxes of infinity. In phase 2, the students were introduced to the grossone methodology after which they were asked to resolve the same paradoxes using this methodology in phase 3. After being presented by the normative and the grossone-based resolutions, the students reflected on them in writing.

Results of this study showed that students' intuitive solutions were similar to grossone-based solutions of the three considered paradoxes. Therefore, the students accepted those solutions.

This study shows the relevance of using an alternative method, which is the grossone methodology, in order to simplify resolutions of some paradoxical situations with infinity for undergraduate students.

**Keywords:** APOS theory, grossone methodology, Infinity, paradoxes of infinity.

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# РАЗРЕШЕНИЕ СТУДЕНТАМИ НЕКОТОРЫХ ПАРАДОКСОВ БЕСКОНЕЧНОСТИ В ОБЪЕКТИВЕ МЕТОДОЛОГИИ ГРОССУАНА

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## Аннотация

В статье описан опыт применения нового математического языка (методология грассуана) в работе со старшими школьниками. В эксперименте приняли участие одиннадцать учащихся 12 класса средней школы в Ливане. Им было предложено решить классические парадоксы бесконечности двумя способами: традиционным и новым (методология грассуана).

На первом этапе школьники, которые учатся по традиционной программе, предложили интуитивные решения трех парадоксов бесконечности. На втором этапе они изучили новый математический термин (грассуан), после чего (на третьем этапе) разрешили те же парадоксы, используя новую методологию. Затем учащиеся сравнили результаты своих решений письменно.

Результаты этого эксперимента показали, что, используя грассуан, школьники смогли решить парадоксы бесконечности (полностью или частично), а интуитивные решения учащихся были аналогичны решениям трех рассматриваемых парадоксов с применением грассуана (поэтому школьники сочли их убедительными). Новый математический язык показал свою эффективность не только в решении вопросов бесконечности, но и в попытках шагнуть за пределы освоения математических понятий по теории APOS.

Данное исследование показывает актуальность использования альтернативного метода объяснения парадоксов бесконечности, которым является использование грассуана, для школьников и студентов младших курсов.

**Ключевые слова:** теория APOS, методология грассуана, бесконечность, парадоксы бесконечности.

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## 1. Introduction

Paradoxes have occupied the human mind so long and the concept of infinity is strongly correlated to many mathematical paradoxes. Exploring the infinite has led to several paradoxes and, conversely, paradoxes related to infinity have paved the way to exploring infinity.

During the 5th century B.C., the Greek philosopher Zeno presented several paradoxes of infinity. For instance, he proposed that in order to move a certain distance, one has to cut half the distance first. Again, in order to cut half the distance, one has to cut half of it, then half of the remaining distance and so on... which implies that motion is impossible! Another paradox considers a race between the Achilles and the tortoise. The tortoise was given a head start. Zeno argued that the Achilles can never reach the tortoise, since by the time the Achilles reaches the point where the tortoise was, the tortoise will have got further [1].

Many are the studies that highlighted how the human mind deals with paradoxes and analyzed them [2–6]. Some studies went further by considering paradoxes as an educational tool through which infinity would be explored [7, 8]. The aim of this study was to check the efficiency of a new methodology: “grossone methodology” [9] in helping the students to cope with paradoxical situations involving infinite iterations. This required an analysis of students’ responses to certain paradoxes related to infinity by using the grossone methodology and analyzing their responses when traditional methods were used, after which a comparison of their arguments using the two methods was promoted.

To make it clear, this study raised questions of how would twelfth graders deal with paradoxical situations related to infinity? How would they deal with the same situations using the grossone methodology?

## 2. Related literature

Paradoxes of infinity are the core of many studies related to infinity. This could be due to the importance of infinite processes in the undergraduate mathematics curriculums (sequences, series, limits...). E. Dubinsky et al. affirmed the beneficial effect of paradoxes in investigating conceptions of infinity [3, 4]. They claimed that through using the mental mechanisms: *interiorization* and *encapsulation*, one might be able to resolve an infinite iterative process paradox. In a more specific study, E. Dubinsky et al. observed how students think about the “Tennis Ball Problem” which involves coordination of three infinite iteration processes [5]. They provided an APOS analysis of the mental constructions needed to solve the problem. Results of this study showed that only one out of fifteen students was able to provide a correct mathematical solution for this problem.

E. Fischbein [6] pointed out that the problem in Zeno’s paradoxes lies in the spatial interpretation of time and motion which are not infinitely divisible. He showed that the solutions of Zeno’s paradoxes are

psychological rather than mathematical. D. Tirosh [10] considered that the contradictions faced by students regarding infinite sets could raise students’ awareness of the importance of the formal definitions in mathematics. Aligning with her study, N. Movshovitz-Hadar & R. Hadass [7] found out that engaging students in solving paradoxes would develop a cognitive conflict that could result in construction of new cognitive structures. Moreover, they pointed out the importance of paradoxes in provoking discussions which serve in developing students’ mathematical thinking.

A. Mamolo & R. Zazkis [8] used paradoxes as a research tool in order to investigate students’ understanding of infinity before and after instruction on cardinality equivalences. Results showed a slight change in students’ intuitive approach when dealing with contexts related to infinity after instruction. A. Mamolo & R. Zazkis recommended an instructional approach that helps students to separate their intuitions from formal considerations. Moreover, A. Mamolo & R. Zazkis observed students’ appeal to process conceptions of infinity and resistance to the actual one. This result is compatible with the results of numerous studies done on conceptions of infinity.

As it is clear from the literature review, introducing paradoxes to undergraduate students is of great importance. On the other hand, the results also show that most undergraduate students are not able to cope with paradoxical situations even after getting some special instruction. A consequence of such experiences could be negative beliefs of infinity as well as feelings of fear or unease when dealing with infinity [11]. So one would think of trying to approach paradoxes related to infinity by using a different method which involves convincing arguments for undergraduate level students.

## 3. Theoretical background

In this section, we present the theoretical framework in which the results were studied and analyzed.

### 3.1. The APOS theory

The APOS theory, devised by E. Dubinsky & M. A. McDonald [12], is a constructive theory of learning that is based on Piaget’s theory of reflective abstraction. It is suitable for undergraduate level studies and provides a deep insight on the formation of mathematical concepts in individuals’ minds. According to E. Dubinsky & M. A. McDonald, when an individual is faced with a problem situation, s/he uses mental mechanisms like *interiorization*, *coordination*, and *encapsulation* in order to construct mental structures that help in making sense of the situation. These mental structures are termed as actions, processes, objects, and schemas. Actions refer to the transformations that are triggered by memory or step-by-step instruction. Processes require transformations that take place in the mind without explicit execution of the steps. When the process is viewed as a totality in which transformations can be applied to it, the individual will have reached the object

level. This happens through encapsulating the process into a cognitive object. The collection of the actions, processes, and objects of a certain mathematical concept form a schema which serves as a framework that can be used in problem situations involving this mathematical concept [12].

E. Dubinsky et al. [4] suggested an APOS analysis of the concept of infinity. They proposed that a process thinking of the infinite refers to an understanding of potential infinity. This is the case when an individual interiorizes infinity to a process that is endless without carrying out the steps explicitly. On the other hand, they related an object thinking of infinity to the understanding of actual infinity. In this case, the individual should have the ability to envision the whole process as a totality and encapsulate it to a cognitive object.

### 3.2. A brief introduction to the Arithmetic of Infinity

The *Arithmetic of Infinity*, introduced by Yaroslav Sergeyev\* in 2003 [9], provides a new way of dealing with infinity and the infinitesimals. The main principle of this methodology is the separation between the mathematical objects and the numeral systems representing them [9, 13, 14]. Unlike the traditional view which considers that the complications related to infinity are a result of its abstract nature, Ya. Sergeyev argues that it is a result of our inability to distinguish between its different sizes [14]. He claims that if we use a sharper lens which allows us to see these distinctions, some complications related to infinity will vanish.

In order to illustrate his idea, Ya. Sergeyev considers a numeral system used by an Amazonian tribe called “Pirahã” [15]. The Pirahãs use a poor numeral system for counting which consists only of: one, two, and many. By using this weak numeral system, they are unable to distinguish, make comparisons or perform computations for quantities that are greater than two. Analogously, the numeral system we work with, is suitable at the level of finite numbers, but does not allow distinctions between different infinities and infinitesimals. For this reason, we need a sharper lens to allow these distinctions. In order to exemplify this analogy, let us exchange the ‘many’ of the Pirahã’s with “ $\infty$ ”.

For instance,

“many” + 1 = “many”, “many” + 2 = “many”,  
“many” + “many” = “many”.

By exchanging “many” with “ $\infty$ ”, we arrive at relations that are used in the traditional calculus:

$$\infty + 1 = \infty, \infty + 2 = \infty, \infty + \infty = \infty.$$

The situation is similar when it comes to indeterminate forms:

“many” – “many” is indeterminate since there is no fixed value of the “many”.

By exchanging “many” with “ $\infty$ ”, we reach at the indeterminate form: “ $\infty$ ” – “ $\infty$ ”.

The *Arithmetic of Infinity* introduces a new infinite unit called “grossone” and denoted by  $\textcircled{1}$ .

$\textcircled{1}$  is the number of elements of the set of natural numbers, hence,  $\forall n \in \mathbb{N}, n \leq \textcircled{1}$ .

$\mathbb{N} = \{1, 2, 3, \dots, \textcircled{1} - 3, \textcircled{1} - 2, \textcircled{1} - 1, \textcircled{1}\}$  where  $\textcircled{1} - 3, \textcircled{1} - 2, \textcircled{1} - 1$ , and  $\textcircled{1}$  are infinite numbers that belong to the set of natural numbers.

$\textcircled{1} + 1, \textcircled{1} + 2, \dots$  are infinite numbers that are not elements in set  $\mathbb{N}$ .

An extended set of natural numbers is as follows:

$$\hat{\mathbb{N}} = \{1, 2, 3, \dots, \textcircled{1} - 1, \textcircled{1}, \textcircled{1} + 1, \textcircled{1} + 2, \dots, 2\textcircled{1}, 2\textcircled{1} + 1, \dots, \textcircled{1}^2 - 1, \textcircled{1}^2, \textcircled{1}^2 + 1, \dots\}$$

The infinitesimals are the inverses of  $\textcircled{1}$ -based infinite numbers.

It is postulated that the arithmetical properties that hold for finite numbers hold for  $\textcircled{1}$ -based numbers [13].

An important contribution to the  $\textcircled{1}$ -based methodology is that it allows representing infinite and infinitesimal numbers in a positional numeral system with infinite radix which is an extension of the system in base ten. Therefore, writing a number in base  $\textcircled{1}$  could be done in an analogous way to that of base 10. For instance,  $2\textcircled{1} + 1$  corresponds to the base  $\textcircled{1}$  record:

$$2\textcircled{1}^1 + \textcircled{1}^0 [13].$$

In addition, using the grossone numeral system, one can perform computations involving finite, infinite and infinitesimal quantities providing accurate results which is not the case when using traditional methods. For instance [13],

$$3(\textcircled{1} - 1) - 2(\textcircled{1} + 4) = 3\textcircled{1} - 3 - 2\textcircled{1} - 8 = \textcircled{1} - 11.$$

We point out that a computing software especially designed for working numerically with grossone-based numbers, has been developed. It is called “*the Infinity Computer*” [14, 17].

Furthermore, using this methodology would eliminate many indeterminate forms and paradoxical situations some of which will be shown in this paper\*\*.

In order to clarify his idea, Ya. Sergeyev presents an example of counting the number of grains in a granary [14]. Due to our inability to count the grains one by one, people use sacks and fill them with seeds and count the number of sacks (it is supposed that all sacks can contain the same unknown number of seeds). If there are too many sacks, and we are not able to count using only elementary units of counting (seeds), motor lorries

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\*\* For detailed information visit: <https://theinfinitycomputer.com>

or train wagons are used (again, it is supposed that all motor lorries can contain the same unknown number of sacks, etc.). Though the number of seeds in each sack is unknown, yet we can register any change to the nearest one seed. For instance, the granary contains 12 wagons, 49 trucks, 56 sacks and 123 seeds. If we add one seed or remove 2 trucks for example, we can detect the increment or decrement. The example of the granary refers to a finite situation in which we are not able to count. Nonetheless, we can extrapolate the idea of introducing an unknown yet fixed value to the infinite case. Linking with ①, we can say that the set of natural numbers is a sack that contains ① elements.

It should be noted that this methodology has been widely applied in many branches of mathematics and computer sciences [9, 13, 14, 18–30]. Several studies showed that this methodology is consistent and non-contradictory [31, 32]. Moreover, this method has proved its efficiency in several pedagogical studies [11, 33–35, 36].

### 3.3. Paradoxes of infinity

In this section, we consider three well-known paradoxes of infinity: Galileo's paradox, Hilbert Grand Hotel paradox and the Ping-pong ball conundrum. The paradoxes are presented. Their resolutions are presented in two ways: the traditional methods and the grossone method.

#### 3.3.1. Galileo's paradox

Galileo's paradox refers to the bijection established between the set of natural numbers and its squares. Of course, this result is paradoxical since for any natural number, one can associate a corresponding square, yet on the other hand, there are more natural numbers than squares. A similar paradoxical situation arises when establishing a bijection between the set of natural numbers and the set of even numbers [28].

1	2	3	4	...
↑	↑	↑	↑	
2	4	6	8	...

In this study, we will consider comparing the set of natural numbers and that of even numbers. According to the traditional set theory, both sets are countable of the same size, which is  $\aleph_0$ .

Now we present the situation from the grossone point of view:

We recall that the number of elements in set  $\mathbb{N}$  is ①.

$$\mathbb{N} = \{1, 2, 3, 4, \dots, \frac{①}{2} - 1, \frac{①}{2}, \frac{①}{2} + 1, \dots, ①\}$$

The set of even numbers:

$$\mathbb{E} = \{2, 4, 6, 8, \dots, \frac{①}{2} - 2, \frac{①}{2}\}$$

1	2	3	4	...	$\frac{①}{2} - 1$	$\frac{①}{2}$
↑	↑	↑	↑		↑	↑
2	4	6	8	...	① - 2	①

As illustrated above the count of the even numbers is  $\frac{①}{2}$ . Comparing both methods, one can see that the *Arithmetic of Infinity* allows expressing infinite numbers, which gives an insight of the last element in both sets and thus allows an accurate comparison of the two sets. According to Ya. Sergeyev [28], this result doesn't contradict the traditional result; it only uses a sharper lens to envision the situation. Analogously, the Pirahãs would say that a set of 5 objects and another of 10 objects are both of the same size, which is "many". This is a correct but imprecise answer.

#### 3.3.2. The Hilbert Grand Hotel paradox

*A grand hotel that has infinitely many rooms has no vacancy. How can a new guest be accommodated in this hotel knowing that only one person is allowed per room?*

One of the traditional ways to solve this paradox is to have the person in room 1 move to room 2 and the person in room 2 to move to room 3 and so on. This can be done since the hotel has infinitely many rooms. In this manner, room number 1 will be available for the new guest. In an analogous way, the result can be generalized and the Hilbert Grand Hotel can accommodate countably infinite number of new guests.

Now we present the same paradox by using the *Arithmetic of Infinity* methodology. We may suppose that the hotel has ① rooms, since the number of rooms of the hotel is countable. If the guest in room number 1 moves to room number 2 and the guest in room number 2 moves to room number 3, and so on ..., then the guest in room number ① will be out of the hotel since the hotel contains ① rooms. This is parallel to what happens in finite cases and so this paradox is no more a paradox [28]!

It should be noted that the traditional approach and that of the *Arithmetic of Infinity* are not contradictory. Both display the same result (it is possible to give the newcomer room number 1) with the difference that ①-based method goes further by showing what would happen to the guest in the last room, which is not visible traditionally [28].

According to F. Caldarola et al. [30], Hilbert's paradox of the Grand Hotel brings about many *semantic problems* and queries of a *logical nature*. In their debate, they considered as an example the word "full" which means in the common used language that there is no possibility to add more. Approaching this same paradox by using grossone methodology, such conflictions no more exist since if the hotel is filled with grossone guests, then no more guests can be accommodated.

#### 3.3.3. The Ping-Pong ball conundrum

*Consider having an infinite set of Ping-Pong balls that are numbered 1, 2, 3, .... You have to place the first ten balls in a huge barrel and then remove ball number 1. Next, you have to place balls numbered 11 to twenty in the barrel and then remove ball number 2. The same procedure is to be repeated over and over again. At the*



end of the experiment how many Ping-Pong balls are left in the barrel?

In this paradox there are two infinite iterations taking place: the balls going into the barrel and the balls going out of the barrel. Since the balls are numbered then both sets of in-going and out-going balls can be put into a one-to-one correspondence with the set of natural numbers. These correspondences assure that at the end of the experiment, the barrel would be empty. The paradox lies in that there are more balls going in than there are balls going out, yet at the end of the experiment the barrel is empty!

Resolving this paradox using the ①-based method, there are ① Ping-Pong balls (the balls are numbered). In every step of the experiment, there are 9 balls in and one ball out of the barrel. This means that the whole procedure, will be done in ①/10 stages. It is clear that at the end of the experiment, there will be 9①/10 balls in the barrel and ①/10 balls out [16].

The grossone-based solution enables us to know at the last stage, the number of balls inside and outside the barrel in terms of ①. This result is compatible with computations done with finite numbers, so the paradox no more exists!

It is noted that the original version of the paradox requires the experiment to be done in exactly one minute, starting from 30 seconds for the first stage, and considering half of the remaining time for each of the following stages successively. But in order to make the paradox simpler, we have chosen not to consider the time iterations.

## 4. Methodology

In this section, we describe the methods used in this study as well as the sample and means of analyzing the results.

### 4.1. Sample and setting

A group of 11 grade 12 students (general sciences section\*) were chosen arbitrary from an official secondary school in Lebanon (Ain-Zahalta). The study was implemented at the beginning of the academic year 2021/2022. General sciences students were chosen for the study, since they get exposed to topics related to infinity in classical methods (limits, sequences, improper integrals...)\*\*. But since the study was done at the beginning of the year, students' direct experiences with infinity were restricted to study of limits of functions.

### 4.2. Method

In the first phase of the study, the students were introduced to each of the paradoxes (Galileo's paradox, Hilbert's Grand Hotel and the Ping-pong ball conundrum) in the traditional way, without any mention of grossone. The paradoxes were presented in increasing order of complexity. After that, the students were

asked to work individually and write down their arguments. For each paradox, after they handed in their solutions, the normative solution was presented, and the students were asked to jot down their reflections on it.

Phase 2 of the study involved an introductory session (an hour) of "The Arithmetic of Infinity"\*\*\*. The methodology was explained in brief since teaching its details are out of the scope of this study. The instruction included introducing the grossone as the number of elements of the set of natural numbers. The illustrations and examples provided by Ya. Sergeyev (Pirahās' numeral system and the granary example) were explained to the students. In addition, the instruction included writing infinite numbers in grossone-based system and performing basic arithmetical operations with infinite numbers and infinitesimals. Particularly, the *Arithmetic of Infinity* was explained in the same way as it was introduced in this paper (see 3.2).

In the third phase, the students resolved the three paradoxes used in phase 1 but this time using the new methodology introduced to them. Each student provided a written response to the paradoxes.

For each paradox, the solution using grossone methodology was presented and again the students wrote their reflections individually.

As an illustration of the Hilbert's Grand hotel paradox, an interesting animation of it was played\*\*\*\*.

The data of this study was collected from the written responses provided by the students before and after being exposed to the new methodology.

## 5. Results

In this section we present and analyze students' responses to the three paradoxes before and after being introduced to the grossone methodology, as well as their reflections on both the normative solutions of the paradoxes and then the grossone-based solutions.

### 5.1. Results by using traditional methods

#### 5.1.1. Results of Galileo's paradox

It was no surprise that 10 out of 11 students considered that there are fewer even numbers than there are natural numbers, specifying that it is equal to half. The students used part-whole methods in their arguments. Some of them even used a finite example as evidence and extrapolated the result to the infinite case. One of the students drew a Venn diagram showing that the set of even numbers is embedded inside the set  $\mathbb{N}$ .

One student wrote: " $\mathbb{N} = \mathbb{E} + \mathbb{O} \Leftrightarrow \mathbb{E} = \mathbb{N} - \mathbb{O}$ , but  $\mathbb{E} = \mathbb{O}$  so  $\mathbb{E} = \frac{\mathbb{N}}{2}$ , where  $\mathbb{E}$  represents the number of evens,  $\mathbb{O}$  the number of odds", — which shows that the intuitive thinking of sizes of infinite sets is similar to that used in finite cases.

\* General sciences section refers to mathematics-majored class in the Lebanese curriculum.

\*\* For more details, check: <http://www.crdp.org>

\*\*\* The content of the Introduction to the method was simulated from: <http://www.numericalinfinities.com>

\*\*\*\* Animation available at: <http://theinfinitycomputer.com>

Only one student considered that the two sets have the same number of elements. She wrote: “...although there are missing numbers in the set of even numbers like 3,5,7... but numbers are so many, they don’t end. We can keep going on and on. So we can say that the two sets are equal in number of elements but their values are different”.

This student shows a conception of the process of adding more and more elements to both sets at the same time endlessly. This answer shows that the student visualizes an endless process, having a conception that infinity is of one size so if the two sets are infinite, they should be equal in size. This conception was termed by F. M. Singer & C. Voica [37] as “single infinity”.

### 5.1.2. Results of Hilbert's Grand Hotel paradox

A common reaction among all the students was criticizing the paradox. Most students argued that there can’t be infinite number of people since the people on earth are no more than 7,8 billion, so the new guest will have a room for sure. It was noticed that though the students criticized the idea of infinite people, they didn’t find the idea of infinite hotel impossible. This result is compatible with that found in [8]. Only one student pointed out this idea. He wrote: “An infinite hotel is impossible since the rooms will reach a size bigger than the earth itself. Maybe they could make layers on top, but it would exceed the earth’s atmosphere and there will be no oxygen on the higher rooms”.

Some students found the idea that a hotel is infinite and full at the same time very conflicting. They argued that if the hotel is infinite, then it can’t be full. This shows the dominance of the process thinking of infinity that most students possess. They cannot envision the hotel as an entity in order to think of it as full or not. APOSly talking, encapsulating the process is a necessary step that enables one to figure out the wholeness of the process which yields to an object in which actions and other processes could be applied.

Aligning with the results in [8], most students gave non-mathematical arguments. One student argued that the rooms could be not available due to maintenance issues or they may be still under construction. Another student suggested dividing a big room to two.

The process thinking (which refers to process level of APOS) was central in almost all the arguments provided by the students. We present some of them:

- “The hotel cannot be full since we can add more and more rooms”;
- “If we want to find a room for the new visitor, it will take days or even years to reach this room”;
- “We know that infinity has no end...”;
- “Infinity doesn’t stop”.

A student argued that if the rooms are infinite then there should be infinite keys so the new guest should be able to get a key. But he was stuck in finding the number of the last room. He wrote: “We don’t know the number of the last room occupied since it is infinite. Infinity

+1 = infinity. We can’t know what happens at infinity”. It is noticeable that this student was the only one who talked about the last room and also the only one who used mathematical arguments in his answer. Despite his attempt to encapsulate the process and view it as a totality (which is an APOS level between the process and the object) (see [38]), the traditional methods he had learned ( $\text{infinity} + 1 = \text{infinity}$ ) were not enough for him to figure out the situation. His sentence: “We can’t know what happens at infinity”, — is an evidence of the need of a lens that is sharper than the one used in the traditional methods.

### 5.1.3. Ping-Pong balls conundrum

Most students approached the problem by considering a finite case and extrapolating it to the case of infinity. This is similar to the way they have approached Galileo’s paradox. For instance, a student wrote: “In the first stage 1 to 10 balls are placed in, then we remove ball number 1. We still have 9 inside the barrel. In stage 2, 18 balls in and we remove ball number 2. So we have 2 balls out. In stage 3, 27 balls in and 3 out... Multiples of 9. Let  $n$  be the number of the balls, then  $n - (n/10)$  are the balls in the barrel. If  $n \rightarrow \infty$ , we get  $\infty - \infty$  which is indeterminate”.

At first, when the student was explicitly writing the number of balls in and those out in each stage, he was thinking at the action level. Then he realized that as the process continues, a multiple of nine balls is inside the barrel without executing each stage, which refers to process level of the APOS theory. The student then was not able to figure out the completeness of the process which yields to the construction of the final object. The last part of his argument shows that the usage of the traditional methods he had learned led to an indeterminate form in which he couldn’t do anything about it. So, he was stuck and stopped at this point. Here again, students’ answers show the need for a more accurate instrument in visualizing infinite numbers and determining the indeterminate cases.

Other answers:

- “ $\infty - \infty/10$ . We cannot count them; it is impossible to finish putting in the balls because they are infinite”;
- “We cannot calculate the exact number because the number of balls is infinite. But we know that the number of balls in the barrel is nine times the number of balls out and the number of balls out equal the stages in which we repeat the experiment”;
- “We cannot determine the number of balls in or those out. They are both infinite, but the infinity of the balls in is larger than the infinity of those out”.

The answers provided by the students share a common idea, that there is infinity of balls inside and outside the barrel, and that the infinite balls inside are much more than those outside the barrel. Some of the students provided a fixed ratio between the balls in and those out. Here the students showed a conception of different sizes of infinity. Moreover, responding that the process

cannot be finished, reveals a process conception and a difficulty in viewing the totality of the whole process.

It is noted that the same student who had a conception of a single infinity regarding Galileo's paradox showed a conception of different sizes of infinity in this paradox! The reason could be the context in which the infinity is presented. In Galileo's paradox, numbers (which are abstract) were the object under study, whereas Ping-pong balls are objects found in real life. The result that the conceptions of infinity are affected by the context in which infinity appears, was also shown in [39].

It is also interesting that the students didn't criticize the infinite number of balls and a barrel that could contain them, unlike their objection about infinite number of guests in the Hilbert Hotel's paradox! This is compatible with the findings in [8].

## 5.2. Results using Arithmetic of Infinity

After the brief introduction of the *Arithmetic of Infinity* methodology, the students were asked to resolve the three paradoxes according to this methodology. An interesting result was that all the 11 students were able to resolve Galileo's paradox correctly. This maybe because the solution of this paradox using grossone is the same as their own solutions in phase 1.

Eight out of eleven students were able to solve correctly the Hilbert's Grand Hotel paradox using grossone methodology. It is noted that the students kept on criticizing this paradox even after the normative and the grossone solutions were provided. They continued to argue about the impossibility of infinite guests. Similarly, in [8], the students found the normative solution impractical and not realistic.

The Ping-Pong ball conundrum is the most complex paradox among the three paradoxes considered since it requires coordination between two infinite iterative processes. Six students out of the 11 were able to provide a correct solution by considering the grossone methodology, which is considered a good result in accordance to its complexity. It is noted that 3 students out of the 6 who gave correct answers provided the number of balls that will be out of the barrel at the end of the experiment, though the number of balls left inside the barrel was required. Maybe those students found it easier to count the number of balls out of the barrel since it is equal to the number of stages performed. But after asking them individually about their response, it was easy for them to tell the number of balls inside the barrel at the end of the experiment as well. So, their answers were considered correct. It was observed that all the students except one, who were able to solve the Ping-Pong ball conundrum were those who solved the Hilbert hotel paradox correctly. This result is no surprise since the paradoxes were presented in increasing order of complexity.

It is noted that, by posing a last element to the set of natural numbers, the *Arithmetic of Infinity* aided the students in thinking of the totality of the set and provide the answer accordingly, rather than focusing on the process that doesn't end.

## 6. Reflections on the solutions of the three paradoxes

After both solutions (the traditional and the grossone-based) were presented, the students reflected on them in writing. Their arguments are summarized in this section.

### 6.1. Galileo's paradox

Ten out of eleven students were not convinced by the normative solution. They insisted on their answers. Some of them used finite examples to show that the solution doesn't work and extended the result to the infinite case. One student used the *proof by contradiction method* by showing that there are elements in  $\mathbb{N}$  which are not in the set of even numbers so they cannot be of same size.

One student used an example to illustrate her idea: *"If we consider that the natural numbers are people and the even numbers chairs. We want to give each person a chair. The number of people is the double of the number of chairs, so we will not be able to seat all the people, even if they are infinite"*.

Another response: *"The problem is that we don't know what happens at infinity so we cannot tell if this solution is right or wrong"*.

The students found the *Arithmetic of Infinity* method convincing, since the answer of the paradox using this method is the same as their intuitive answers. Only one student was still suspicious about posing a last element of set  $\mathbb{N}$ , which is infinite.

### 6.2. Hilbert's Grand Hotel paradox

Nine out of the 11 students were totally convinced by the grossone-based solution. One of the students found both methods not convincing. She argued that the paradox itself doesn't make sense, since an infinite hotel is impossible as well as an infinite number of guests. She also criticized the grossone method for labeling a last room of an infinite hotel.

Another student was not convinced by the grossone method. According to him, not only one guest can be accommodated but also 10 million, since infinity has no end and talking about a last room is not possible. This student neglected the fact that the hotel is full in his reasoning and still show a conception of a non-ending process. But this same student was convinced by the grossone-based solution of Galileo's paradox! This asserts that students' conceptions are altered according to the context in which infinity is considered.

### 6.3. Ping-pong ball conundrum

As it is shown above, the students' intuitive solution of this paradox is similar to the grossone-based solution. Almost all the students were convinced by it. Even the students who were not able to provide the correct solution in grossone terms, were persuaded by it. Only one student rejected the grossone methodology (same student who rejected it in the two previous paradoxes). But still, this student was not convinced by



the normative solution either. She believes the number of balls in the barrel would be infinite at the end of the experiment but rejected specifying it in the grossone method.

## 7. Conclusions

The results of this paper regarding the traditional methods are similar to those found in previous research on paradoxes [3–5, 8, 39]. For instance; the confusions faced by students, the frustration and escaping to non-mathematical solutions, thinking at the process level and failing to encapsulate the process, showing different conceptions of infinity depending on the given context (mathematical or real life situation), rejection of the normative solutions of paradoxes related to infinity. However, this study added to the literature the effect of the *Arithmetic of Infinity* method on students' approaches, arguments, and solutions to paradoxical cases involving infinity as well as their reflections and points of view to both the normative and the grossone-based resolutions.

Before being introduced to the grossone methodology, grade 12 students were not able to cope with the paradoxes. Their responses show that the broad meaning of the word “infinity” didn't aid them to evaluate the paradoxical situations presented to them. On the other hand, 100 % were able to solve Galileo's paradox, 73 % were able to solve Grand Hotel paradox, and 55 % were able to solve the Ping-pong ball conundrum using the *Arithmetic of Infinity* method. The reason of this result is that the grossone methodology is very similar to students' naïve intuitions and is in line with their own reasoning and responses. Therefore, they found these resolutions convincing, unlike the normative resolutions that were rejected by most of them.

Furthermore, introducing a last element to the set of natural numbers, enabled thinking of the wholeness of the set and helped in conceiving the processes related to it as a totality. By this, the *Arithmetic of Infinity* served in pushing students' conceptions to think beyond the process level of the APOS theory.

As we have seen from the results, the *Arithmetic of Infinity* method had proven its efficiency in eliminating paradoxical situations related to infinity. Further investigations on a wider sample of students and of more advanced mathematical background could be beneficial in confirming our findings.

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