

## WAYS OF USE OF COMPUTER ALGEBRA SYSTEMS IN THE TEACHING OF ADVANCED SECTIONS OF MATHEMATICS

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### **Abstract**

In this article, possible ways of use of computers for the teaching of advanced sections of mathematics that traditionally belong to undergraduate curricula, namely elements of calculus, number theory and abstract algebra are considered. Use of computer technologies can help also to implement such approaches as genetic method and the use of various modes of representation in education. According to cultural-historical theory of L. S. Vygotsky, computer technologies can be considered as the tool for the construction of concepts in the process of learning. The most appropriate for teaching advanced mathematics are such computer algebra systems as Maple, Mathematica, and various systems of dynamic geometry. We will consider the possibilities of Geogebra for the work with functions at the initial stages of undergraduate calculus courses, namely for the work with concepts of limits and derivatives of functions.

**Keywords:** information technologies, computer algebra systems, Geogebra, algebra, number theory, calculus.

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## **Introduction**

Information technologies more and more penetrate mathematics teaching nowadays. In a digital era, when students from early age join digital technologies such as mobile devices (phones, tablets), home computers, the Internet, these technologies, certainly, cannot stay away from the teaching processes, in particular in mathematical education. Experiments on the computer-aided teaching have started more than half a century ago. Computers are used in the teaching of mathematics in many countries, for example, interactive activities with computer technologies are used in mathematics teaching in Singapore [1, 2].

Nevertheless, there are still no reliable confirmations that use of computers in classrooms significantly increases the results of teaching. Moreover, materials of the PISA study of the last decades demonstrated that the best results belonged to just those countries that least used computers in the school teaching. These are the countries of Southeast Asia: Japan, South Korea and China that traditionally show the best achievements in the PISA and TIMSS studies [3]. From this fact, one can make a conclusion that not the use of information technologies itself, but rather effective methods of their use in mathematical education is important for the improvement of results of the teaching.

In this paper we will consider possible ways of use of computers for the teaching of advanced sections of mathematics that traditionally belong to undergraduate curricula: elements of calculus, number theory and abstract algebra.

In the last decades (since about 1984 when the first package of computer algebra Macsyma intended for broad application was published), researchers' interest in use of computer technologies in the tertiary education, in particular, in the preparation of teachers, grows.

Davis, Hersh and Marchisotto [4] indicated the following advantages and shortcomings of use of computers in mathematical courses at the higher school: "(1) the verification of mathematical statements and identities; (2) the discovery of new facts through computer experimentation and induction (not mathematical induction); (3) the great successes and occasional pitfalls of the computation process; (4) the limitations imposed by the digital language as opposed to the richer existential language of "full" mathematics; (5) the extent to which one needs mathematical knowledge and expertise beyond what is built into commercial mathematical software; (6) the appreciation of the computational infrastructure of our civilization, an infrastructure that is often hidden from view".

Some prominent researchers of mathematical education created computer programs for the teaching

advanced sections of mathematics: Tall [5]; Li & Tall [6], Yerushalmi & Schwartz [7]; Yerushalmi & Shternberg [8], Kaput & Rochelle [9], Dubinsky & Leron [10]. Most interesting looked attempts to teach using computers the topics, extremely difficult for traditional teaching, for example, abstract algebra, including group theory [11], and linear algebra [12], and also attempts to use computers for the learning of concepts by students [13].

Nevertheless, the most appropriate for the teaching of mathematics (including its advanced sections) are such powerful universal computer algebra systems as Maple and Mathematica and also various systems of dynamic geometry like Cabri Geometer and Geometer's Sketchpad. Freely distributed computer algebra system GAP is suitable for teaching elements of abstract algebra and mathematical logic.

In the last decades the leading positions in using for teaching elementary and college algebra, geometry and elements of calculus has been occupied by Geogebra application.

## Theoretical framework

The theoretical framework of our research is the cultural-historical theory of L. S. Vygotsky [14] and also the activity theory according to A. N. Leontyev [15, 16].

Use of computer technologies can help also to implement such approaches as genetic method [17–19] and the use of various modes of representation in education [20, 21].

According to cultural-historical theory, computer technologies can be considered as the tool for the construction of concepts in the process of learning.

According to the theory of A. N. Leontyev [15, p. 208–212], “any substantial activity answers a need materialized in a motive; its main generators are the purposes and corresponding actions... The task is just a purpose given in certain conditions... By operations we mean ways of realization of actions... Actions... are correlated to purposes and operation to conditions... the genesis of an action lays in the exchange of activities and derives from the intrapsychologization of them. Every operation is a result of transformation of an action occurring as a result of its inclusion in another action and its subsequent “mechanization”... Actions are processes subordinate to the conscious purposes... the operations... directly depend on conditions of achievement of a concrete purpose”.

A. N. Leontyev [16] argued that actions on learning concepts, as well as any actions, consist of operations, which are almost unconscious or completely unconscious. These operations are essentially “contracted” actions with the concepts of the previous level of abstraction. As M. A. Kholodnaya [22] noted, “a contraction is immediate reorganization of the complete set of all available... knowledge about the given concept and transformation of that set into a generalized cognitive structure”. Close to the Soviet conceptions of actions and operations as contracted actions in mathematics teaching is the APOS theory of E. Dubinsky [23].

## Use of computer algebra systems in the teaching of advanced sections of mathematics

Some types of information technologies already became habitual in the process of undergraduate education in Russian Federation: for example, pocket calculators, including scientific, graphical and programmable ones. Teachers should not only take this fact into account but also actively use calculators as the additional tool for improving perception of a subject by students. Use of pocket calculators in the undergraduate educational process exempts students from routine calculations freeing their attention for substantial problems. Certainly, it is possible to use pocket calculators when performing bulky calculations like those in the Horner's scheme for polynomials.

Powerful systems of computer algebra, for example, Maple and Mathematica (and also Wolfram Alpha online platform), can be effectively used in tertiary mathematical education. These applications facilitate calculations with complex numbers, including approximate calculations and calculation of roots up to the sixth degree. In linear algebra, it is possible to solve systems of linear equations, calculate determinants, and find eigenvalues and eigenvectors. Furthermore, one can factorize polynomials and integers etc.

Very useful both for deepening and consolidation of knowledge, and for the development of mathematical thinking of students, by our experience, is students' independent construction of computer programs realizing algorithms found both in the main curriculum and in additional parts, for example, in research project assignments.

Certainly, performance of such tasks by students within research projects appears more real. Recently, students under our supervision have written programs realizing following algorithms: linear representation of the greatest common divisor (GCD) of two numbers; decomposition of a rational number in the continued fraction and calculation of its convergents; solving linear congruences and linear Diophantine equations with two unknowns using continued fractions; continued fraction expansions of quadratic irrationals; calculation of values of quadratic irrationals by their continued fraction expansions; the “Chinese algorithms” (i. e. algorithms based on the Chinese remainder theorem) for integers and polynomials; construction of groups of congruence classes and group of permutations, checking their commutativity, search of their subgroups, checking the normality of subgroups; finding rational roots of a polynomial; solving equations of the 3rd and 4th degree; representation of a symmetric polynomial as a polynomial expression in elementary symmetric polynomials etc.

In the articles [20] and [24] we already described how one can effectively use Geogebra for the introduction of the function. Now we will consider the possibilities of Geogebra for the work with functions at the initial stages of undergraduate calculus courses, namely for the work with concepts of limits of functions, derivatives and the continuity of function.

The examples are taken from the textbook “Algebra and Elements of Calculus” by Mordkovich [25]. In this textbook, the concept of the limit of a function at the point is only described as the explanation of the notation

$$\lim_{x \rightarrow a} f(x) = b:$$

“...If values of the argument are chosen closer and closer to the value  $x = a$ , then values of function less and less differ from  $b$  ... At the same time, emphasize that the point  $x = a$  itself is excluded from consideration” [25, p. 150].

Using Geogebra we will make this explanation visual and we will introduce a formal concept of the limit of a function at the point, using computer technologies as the tool in the learning activity of students.

Consider the function  $f(x) = x^2$  and chose on the axis  $Ox$  the point with  $x$ -coordinate  $x_0 = 2$  and two points  $A$  and  $B$  standing at the distance  $\delta$  (changing in this case from 0 to 1) from  $x_0$  to the left and to the right respectively. The movement of the point  $A$  illustrates approaching  $x_0$  from the left, and the movement of the point  $B$  illustrates approaching  $x_0$  from the right. Students are asked to analyze what will happen to the values of the function on  $x$ -coordinates of  $A$  and  $B$  when the slider  $\delta$  moves from 1 to 0, to write down the results of the observations and to make a conclusion that the closer  $x$ -coordinates of  $A$  and  $B$  approach  $x_0$ , less the values of function on these numbers differs from 4, independently of the side of the approach (fig. 1, 2).

Thus, this task demonstrates that the value  $f(2) = 4$  of the function is the limit. Thus, the limit of the function  $f(x) = x^2$  as  $x$  approaches 2 is equal to 4:

$$\lim_{x \rightarrow 2} x^2 = 4$$

It is possible to consider also any other known continuous function later and find the value of the limit at the chosen point.

Through the work with this applet actions of students are integrated into processes, and the limit does not seem to students something abstract, but rather the concrete concept thanks to geometrical interpretation in Geogebra.

After that we consider the limit of the function  $f(x) = x^2 + 1$  at the point using the  $(\varepsilon, \delta)$ -definition:

We say that the number  $A$  is the limit of the function  $f(x)$  at the point  $x_0$  if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that for all  $x$  such that  $|x - x_0| < \delta$  the inequality  $|f(x) - A| < \varepsilon$  holds.

For this purpose, students are offered to work with the following applet in which it is possible to change both  $x_0$  and values of  $\varepsilon$  and  $\delta$  (fig. 3).

After studying limits and before introducing a concept of derivative function, authors of the textbook suggest considering the tasks leading to the concept of a derivative [24, p. 157]. Two examples are given: one about the speed of the movement and the second about a tangent to the graph of a function. We will consider the solution of the second task using Geogebra program, not only having calculated the slope of a tangent as it

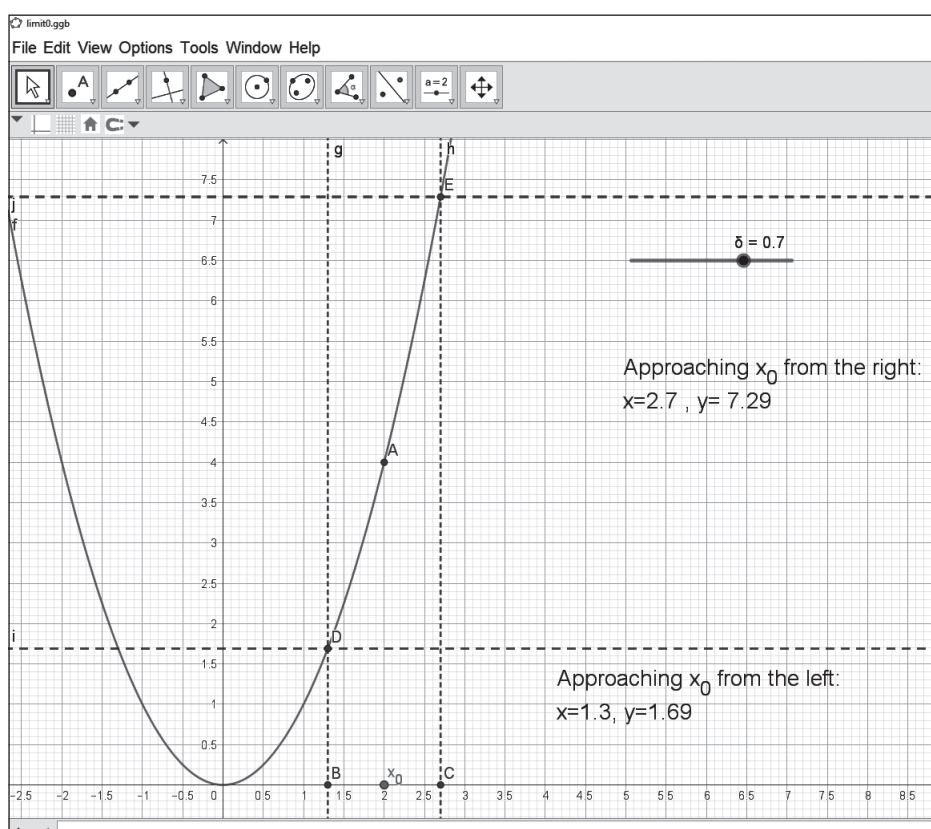


Fig. 1

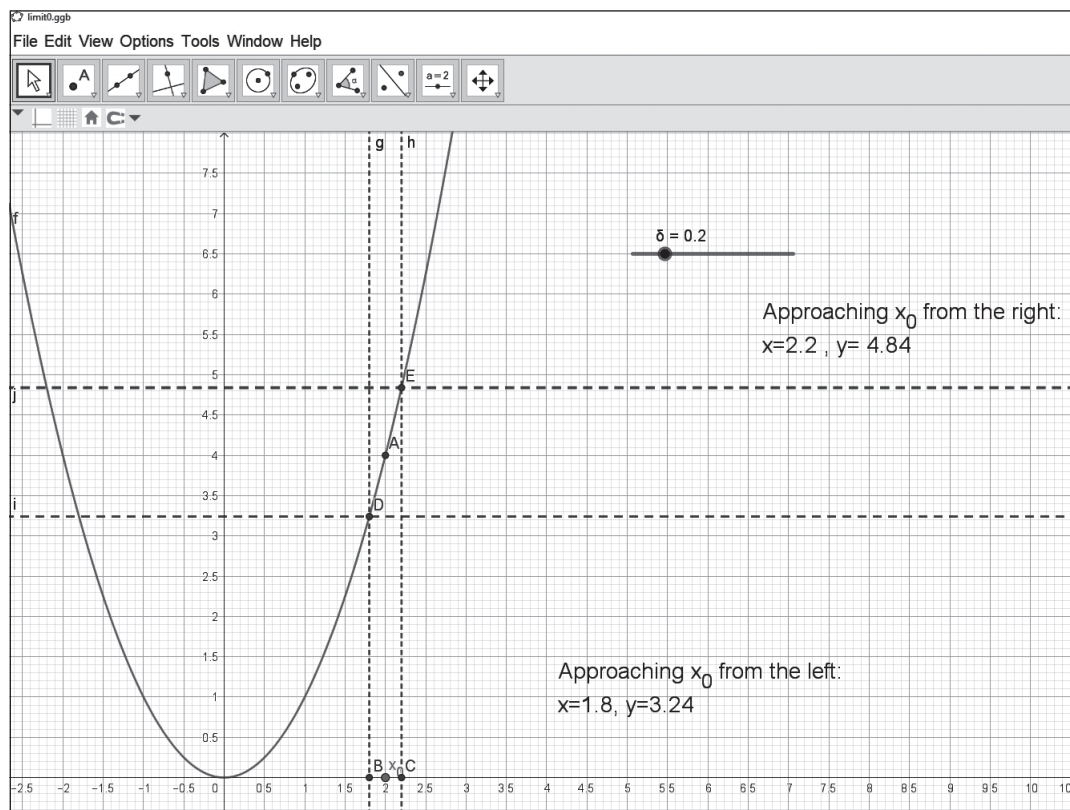


Fig. 2

is asked by the task, but also tracking the interrelation between a tangent and a secant.

**Problem:** On the graph of the function  $f(x)$  the point is chosen. Find the slope of the tangent at this point [24, p. 158].

Consider the function and chose the points  $A$  and  $B$  on its graph (fig. 4). We construct a secant through

these points and also the tangent at  $A$ . Moving the point  $B$  closer to  $A$  using the slider  $h = \Delta x$ , we see that the tangent is the limit state of the constructed secant. The program allows also calculating slopes of the secant and the tangent.

After that it is possible to introduce the formal definition of a derivative.

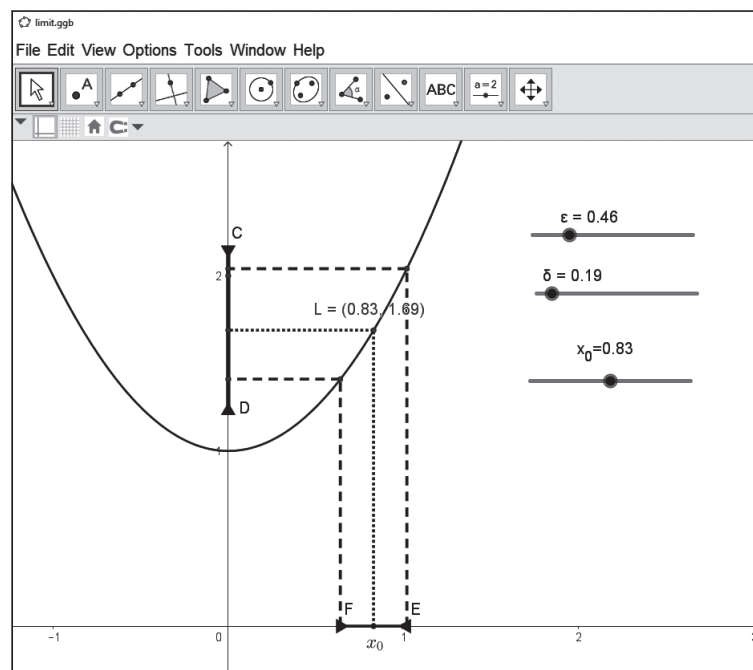


Fig. 3



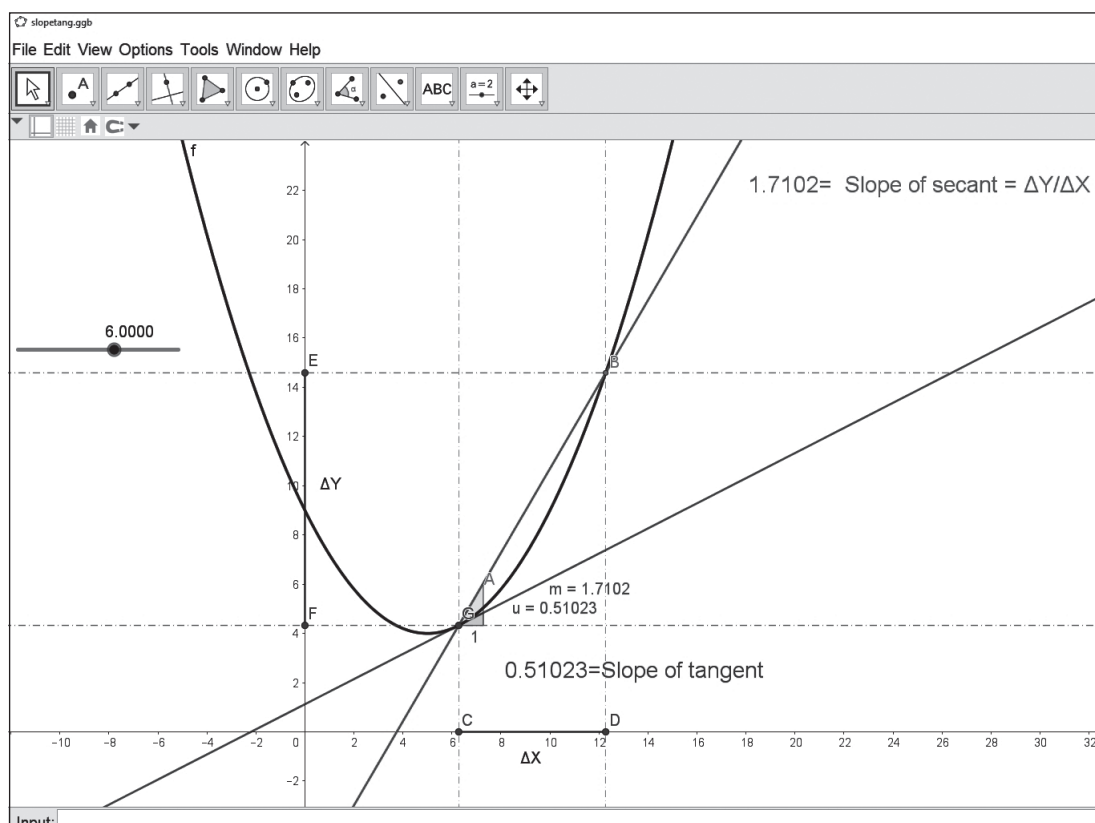


Fig. 4

## Conclusion

Our examples show how computer technologies can help to realize the activity approach in the teaching of advanced sections of mathematics, promoting understanding and deep acquisition of theoretical concepts. The experience of implementation of the constructions and applets described in this article show that they strongly improve the knowledge of students as well as their attitude to advanced sections of mathematics.

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## ВАРИАНТЫ ПРИМЕНЕНИЯ СИСТЕМ КОМПЬЮТЕРНОЙ АЛГЕБРЫ В ОБУЧЕНИИ УГЛУБЛЕННЫМ РАЗДЕЛАМ МАТЕМАТИКИ

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### Аннотация

В статье рассматриваются возможные пути использования компьютеров для обучения сложным разделам математики, традиционно относящимся к программам высшей школы: элементам математического анализа, элементам теории чисел и абстрактной алгебры. Использование компьютерных технологий может помочь также осуществить такие методы, как генетический подход и использование различных способов представления изучаемых объектов в образовании. Согласно культурно-исторической теории Л. С. Выготского, компьютерные технологии можно рассматривать как инструмент для формирования понятий в процессе обучения. Наиболее приспособлены для обучения высшей математике такие системы компьютерной алгебры, как Maple и Mathematica, а также различные системы динамической геометрии. Показаны возможности программы Geogebra для работы с функциями на начальных этапах вузовских курсов математического анализа, в частности, для работы с понятиями пределов и производных функций.

**Ключевые слова:** информационные технологии, системы компьютерной алгебры, Geogebra, алгебра, теория чисел, математический анализ.

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